数理解析研究所講究録
955 巻 1996 年 75-79

The number of compact leaves of a one-dimensional foliation on the $2n-1$ dimensional sphere S^{2n-1} associated with a holomorphic vector field.

Toshikazu Ito

Introduction

Let $Z=\sum_{i=1}^{n} f_i(z)\partial/\partial z_i$ be a holomorphic vector field in some neighborhood of the $2n$-dimensional closed disk $\bar{D}^{2n}(1)=\{ z\in \mathbf{C}^n \mid \| z \|\leq 1 \}$ in $\mathbf{C}^n$. We denote by $\mathcal{F}(Z)$ the foliation defined by the solutions of Z. In this paper we will prove the following

THEOREM A. *If the $2n-1$ dimensional sphere $S^{2n-1}(1)$, which is the boundary $\partial\bar{D}^{2n}(1)$ of $\bar{D}^{2n}(1)$, is transverse to $\mathcal{F}(Z)$ then the number of the compact leaves of the foliation $\mathcal{F}(Z)|_{S^{2n-1}(1)}$ is $1,2,\ldots,n$ or ∞.*

In [5], A. Douady and the author proved the following Poincaré-Bendixson type theorem for a holomorphic vector field.

THEOREM 0.1 (A. Douady and T. Ito). *If $S^{2n-1}(1)$ is transverse to $\mathcal{F}(Z)$, then each leaf L of $\mathcal{F}(Z)$ which crosses $S^{2n-1}(1)$ tends to the unique singular point P of Z in $\bar{D}^{2n}(1)$. Furthermore, since we can move P to the origin $\mathbf{0}$ of $\mathbf{C}^n$ by the Möbius transformation, we see that the sphere $S^{2n-1}(r)=\{ z\in \mathbf{C}^n \mid \| z \|= r \}$ is transverse to $\mathcal{F}(Z)$ for any real number r, $0<r\leq 1$.*

In the case $n=2$ we used Theorem 0.1 as well as the existence theorem of separatrix proved by C. Camacho and P. Sad ([3]) to obtain an affirmative answer to a special case of the Seifert conjecture:

COROLLARY 0.2 ([5]). *Under the hypothesis of Theorem 0.1, the foliation $\mathcal{F}(Z)|_{S^3(1)}$ on $S^3(1)$ has at least one compact leaf.*

We use Theorem 0.1 to prove the following

THEOREM B. *Under the hypothesis of Theorem 0.1, the set of the eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$ of the $n\times n$ matrix $\left(\dfrac{\partial f_i}{\partial z_j}(0)\right)$ belongs to the Poincaré domain.*

This research was partially supported by Grant-in Aid for Scientific Research (C) (NO. 06640181) from the Ministry of Education, Science and Culture, and by the Joint Center for Science and Technology of Ryukoku University.

The proof of Theorem A follows from Theorem 0.1, Theorem B and the Poincaré-Dulac theorem ([6], [4]. See §3).

The author wishes to thank Xavier Gómez-Mont and André Haefliger for their advice.

1. Examples

To shed some light on Theorem A, we give some examples in this section.

EXAMPLE 1.1. Let λ_1 and λ_2 be non-zero complex numbers. Assume that λ_1/λ_2 is not a negative real number. Consider $Z = \lambda_1 z_1 \partial/\partial z_1 + \lambda_2 z_2 \partial/\partial z_2$ on $\mathbf{C}^2$. For any positive real number r, the 3-dimensional sphere $S^3(r)$ is transverse to $\mathcal{F}(Z)$. The solution set L_w of Z with the initial condition $w = (w_1, w_2)$ is $\{ (z_1, z_2) = (w_1 e^{\lambda_1 T}, w_2 e^{\lambda_2 T}) \in \mathbf{C}^2 | T \in \mathbf{C} \}$. In particular, if w_1 is different from zero we may write

$$z_2 = w_2 e^{\lambda_2/\lambda_1 \log(z_1/w_1)}. \tag{1.1}$$

Case (i). If $\lambda_2/\lambda_1 = q/p$ is a positive rational number every leaf of $\mathcal{F}(Z)|_{S^3(1)}$ is compact. This is a Seifert fibration over $S^3(1)$. In the case where λ_2/λ_1 is equal to 1, $\mathcal{F}(Z)|_{S^3(1)}$ is a Hopf fibration. In this case we have infinitely many compact leaves.
Case (ii). If λ_2/λ_1 is either positive irrational or non-real, then $\{ (z_1, 0) \in \mathbf{C}^2 | |z_1| = 1 \}$ and $\{ (0, z_2) \in \mathbf{C}^2 | |z_2| = 1 \}$ are compact leaves of $\mathcal{F}(Z)|_{S^3(1)}$. The equation (1.1) implies that the set $L_w \cap S^3(1)$ is not a compact leaf when every w_i is different from zero. In this case $\mathcal{F}(Z)|_{S^3(1)}$ has exactly two compact leaves.

EXAMPLE 1.2. Let λ and ϵ be two non-zero complex numbers. Consider $Z = \lambda z_1 \partial/\partial z_1 + (\lambda z_2 + \epsilon z_1)\partial/\partial z_2$. The solution set L_w is $\{ (z_1, z_2) = (w_1 e^{\lambda T}, (w_2 + \epsilon w_1 T)e^{\lambda T}) | T \in \mathbf{C} \}$. If w_1 is different from zero we may write

$$z_2 = \left(w_2 + \frac{\epsilon w_1}{\lambda} \log\left(\frac{z_1}{w_1}\right)\right)\left(\frac{z_1}{w_1}\right). \tag{1.2}$$

If $r > 0$ is small $S^3(r)$ is transverse to $\mathcal{F}(Z)$. If $r > 0$ is large, on the other hand, $S^3(r)$ is not transverse to $\mathcal{F}(Z)$. In the case where $S^3(r)$ is transverse to $\mathcal{F}(Z)$, the set $\{ (0, z_2) \in \mathbf{C}^2 | |z_2| = r \}$ is a compact leaf of $\mathcal{F}(Z)$. The equation (1.2) implies that the leaf $L_w \cap S^3(r)$ is not compact if w_1 is different from zero. Thus $\mathcal{F}(Z)|_{S^3(r)}$ has exactly one compact leaf.

EXAMPLE 1.3. Let λ and a be two non-zero complex numbers. Let k be an integer bigger than two. Consider $Z = \lambda z_1 \partial/\partial z_1 + (k\lambda z_2 + a z_1^k)\, \partial/\partial z_2$. The solution set L_w of Z is

$$\begin{aligned} z_1 &= w_1 e^{\lambda T} \qquad \text{and} \\ z_2 &= \left(w_2 + \int_0^T a w_1^k e^{k\lambda T} \cdot e^{-k\lambda T} dT \right) e^{k\lambda T} \\ &= (w_2 + a w_1^k T) e^{k\lambda T}. \end{aligned}$$

If w_1 is different from zero we may write

$$z_2 = \left(w_2 + \frac{a w_1^k}{\lambda} \log(z_1/w_1) \right) \left(\frac{z_1}{w_1}\right)^k. \tag{1.3}$$

For a small $r > o$, $S^3(r)$ is transverse to $\mathcal{F}(Z)$ and the set $\{ (0, z_2) \in \mathbf{C}^2 | \, |z_2| = r \}$ is a compact leaf of $\mathcal{F}(Z)|_{S^3(r)}$. We see from the equation (1.3) that $L_w \cap S^3(r)$ fails to be compact if $w_1 \neq 0$. Thus $\mathcal{F}(Z)|_{S^3(r)}$ has one and only one compact leaf.

We mention that we investigated in ([5]) a global property of contact sets between spheres and $\mathcal{F}(Z)$.

2. The non-existence of transversal maps

Let μ_i $(1 \leq i \leq n)$ be non-zero complex numbers. Assume that the set $\{\mu_1, \dots, \mu_n\}$ belongs to the Siegel domain. Consider a linear vector field $Z = \sum_{i=1}^{n} \mu_i z_i \partial/\partial z_i$ on $\mathbf{C}^n$. To prove Theorem B we need a non-existence theorem of a transversal map f of a manifold to the foliation $\mathcal{F}(Z)$.

THEOREM 2.1. *Let μ_1 and μ_2 be non-zero complex numbers. Consider $Z = \mu_1 z_1 \partial/\partial z_1 + \mu_2 z_2 \partial/\partial z_2$ on $\mathbf{C}^2$. Let M be a closed connected C^∞-manifold of dimension either two or three. If μ_1/μ_2 is a negative real number, then there exists no C^∞-map φ of M to $\mathbf{C}^2$ such that $\varphi(M)$ is transverse to $\mathcal{F}(Z)$.*

PROOF. Suppose that there exists a C^∞-map φ of M to $\mathbf{C}^2$ such that $\varphi(M)$ is transverse to $\mathcal{F}(Z)$. We may select a negative rational number $-p/q$ sufficiently close to μ_1/μ_2 such that $\varphi(M)$ is transverse to $\mathcal{F}(Z')$, where Z' is the linear vector field defined by $Z' = pz_1\partial/\partial z_1 - qz_2\partial/\partial z_2$. The solution L_w of Z' with the initial point $w = (w_1, w_2)$ is $z_1^q z_2^p = w_1^q w_2^p$. Set $F(z_1, z_2) = z_1^q z_2^p$. Then the map $\Phi = |F \circ \varphi| : M \xrightarrow{\varphi} \mathbf{C}^2 \xrightarrow{F} \mathbf{C} \xrightarrow{|\cdot|} \mathbf{R}$ attains a maximal value $\Phi(P)$ at some point $P \in M$. At the point $\varphi(P)$, $\varphi(M)$ is not transverse to $\mathcal{F}(Z')$, but this contradicts our transversality assumption. $\square$

THEOREM 2.2. *Consider a linear vector field $Z = \sum_{i=1}^{n} \mu_i z_i \partial/\partial z_i$ on $\mathbf{C}^n$, $n \geq 3$, where the μ_i's are non-zero complex numbers and the μ_i/μ_j's, $i \neq j$, are imaginary. Let M be a $2n-2$ or $2n-1$-dimensional closed connected C^∞-manifold. If the set $\{\mu_1, \dots, \mu_n\}$ belongs to the Siegel domain, then there is no C^∞-map φ of M to $\mathbf{C}^n$ such that $\varphi(M)$ is transverse to $\mathcal{F}(Z)$.*

PROOF. Let $\Sigma = \{ z \in \mathbf{C}^n | \sum_{i=1}^{n} \mu_i z_i \bar{z}_i = 0 \}$ be the contact set between the spheres $S^{2n-1}(r)$ and $\mathcal{F}(Z)$. Then the set Σ is a cone and $\Sigma - \{0\}$ is a submanifold of dimension $2n - 2$. C. Camacho, N. H. Kuiper and J. Palis proved the following

Fact ([2]). If we take a point $w \in \Sigma - \{0\}$, the distance between L_w and the origin 0 of $\mathbf{C}^n$ attains a unique minimum at w and $L_w \cap \Sigma = \{w\}$. Further the set $W = \{ z \in \mathbf{C}^n | 0 \notin \bar{L}_z \}$ of leaves which do not tend to 0 is diffeomorphic to $(\Sigma - \{0\}) \times \mathbf{C}$. The projection map $\pi : W \to \Sigma - \{0\}$ indicates that each leaf $L \subset W$ corresponds to the point $L \cap \Sigma$.

Assume that there exists a C^∞-map φ of M to $\mathbf{C}^n$ such that $\varphi(M)$ is transverse to $\mathcal{F}(Z)$. The transversality condition implies that the restricted map $\pi|_{W \cap \varphi(M)} : W \cap \varphi(M) \to \Sigma - \{0\}$ is a submersion. Since $\pi(W \cap \varphi(M))$ is open closed connected in $\Sigma - \{0\}$, $\pi(W \cap \varphi(M))$ is equal to $\Sigma - \{0\}$. This contradicts the fact that $\pi(W \cap \varphi(M))$ is bounded. $\square$

We will conclude this section by proving Theorem B.

PROOF OF THEOREM B. We calculated in [5] that the index of Z at the origin is one. Hence every eigenvalue of the matrix $(\partial f_i/\partial z_j(0))$ is different from zero.

It follows from Theorem 0.1 that for small enough $r_1 > 0$ the linear part $Z^{(1)} = \sum_{i=1}^n(\sum_{j=1}^n \partial f_i/\partial z_j(0)z_j)\partial/\partial z_i$ of Z is transverse to $S^{2n-1}(r_1)$. Suppose that the set $\{\lambda_1, \dots, \lambda_n\}$ does not belong to the Poincaré domain. We may choose an $n \times n$ matrix $A = (a_{ij})$ close enough to $(\partial f_i/\partial z_j(0))$ that the set of the eigenvalues of A satisfies the conditions of Theorem 2.1 or Theorem 2.2. The sphere $S^{2n-1}(r_1)$ is transverse to $\mathcal{F}(\tilde{Z}^{(1)})$, where $\tilde{Z}^{(1)}$ is the linear vector field defined by $\tilde{Z}^{(1)} = \sum_{i=1}^n(\sum_{j=1}^n a_{ij}z_j)\partial/\partial z_i$. This is a contradiction to Theorem 2.1 or Theorem 2.2. □

3. Proof of Theorem A

We recall first a theorem due to H. Poincaré ([6]) and H. Dulac ([4]), which we shall call the Poincaré-Dulac linearization and polynomialization at an isolated singular point of a holomorphic vector field.

Let $Z = \sum_{i=1}^n f_i(z)\partial/\partial z_i$ be a holomorphic vector field defined on some neighborhood of the origin 0 of $\mathbf{C}^n$. Assume that the origin is an isolated singular point of Z.

THEOREM 3.1 (H. Poincaré and H. Dulac). *If the set of eigenvalues of the matrix $(\partial f_i/\partial z_j(0))$ belongs to the Poincaré domain, then there exists a biholomorphic map Φ of some neighborhood of* 0 *to another neighborhood of* 0 *in* $\mathbf{C}^n$, $\Phi(z) = w$, $\Phi(0) = 0$, *such that* $\Phi_* Z = W$ *with*

$$W = \lambda_1 w_1 \partial/\partial w_1 + \sum_{i=2}^n (\lambda_i w_i + b_i w_{i-1} + P_i(w_1, \dots, w_{i-1}))\partial/\partial w_i,$$

where the b_i's are either 0 *or* 1 *defined by the Jordan block of $(\partial f_i/\partial z_j(0))$ and the $P_i(w_1, \dots, w_{i-1})$'s are polynomials defined as follows:*
Let $m_i = (m_i(1), \dots, m_i(i-1))$ be an $(i-1)$-tuples of non-negative integers such that $\sum_{k=1}^{i-1} m_i(k) \geq 2$ and $\lambda_i = \sum_{k=1}^{i-1} m_i(k)\lambda_k$. Define P_i by $P_i(w_1, \dots, w_{i-1}) = \sum_{m_i} a_{m_i} w_1^{m_i(1)} \cdots, w_{i-1}^{m_i(i-1)}$. Here the a_{m_i} are complex numbers.

We note for example in the case where $n = 2$ the W is one of the following:

1. $W = \lambda_1 w_1 \partial/\partial w_1 + \lambda_2 w_2 \partial/\partial w_2$.
2. $W = \lambda w_1 \partial/\partial w_1 + (\lambda w_2 + w_1)\partial/\partial w_2$.
3. $W = \lambda w_1 \partial/\partial w_1 + (k\lambda w_2 + a w_1^k)\partial/\partial w_2$.

We are now ready to prove Theorem A.

PROOF OF THEOREM A. We may assume, using the Möbius transformation, that the unique singular point is the origin 0 of $\mathbf{C}^n$. By the grace of Theorem B and Theorem 3.1 we may select a sufficiently small number $r_0 > 0$ so that $\mathcal{F}(Z)|_{\bar{D}^{2n}(r_0)}$ is biholomorphic to $\mathcal{F}(W)|_{\Phi(\bar{D}^{2n}(r_0))}$. Then $\mathcal{F}(Z)|_{S^{2n-1}(r_0)}$ has 1, 2,..., n or infinitely many compact leaves. By Theorem 0.1 $\mathcal{F}(Z)|_{S^{2n-1}(r_0)}$ is C^ω-diffeomorphic to $\mathcal{F}(Z)|_{S^{2n-1}(1)}$. This completes the proof of Theorem A. □

REMARK. M. Brunella and P. Sad ([1]) proved the following theorem. Define a linear hyperbolic foliation $\mathcal{L}_\lambda$ in $\mathbf{C}^2$ by $xdy + \lambda ydx = 0$, $\lambda \in \mathbf{C} - \mathbf{R}$.

THEOREM (M. Brunella and P. Sad). *Let $\Omega \subset \mathbf{C}^2$ be a generalized bidisc and let $\mathcal{F}$ be a holomorphic foliation defined in a neighborhood of $\overline{\Omega}$ and transverse to $\partial\Omega$. Then there exists a locally injective holomorphic map ϕ which sends a neighborhood of $\overline{\Omega}$ to a neighborhood of* 0 *in* $\mathbf{C}^2$ *and such that* $\mathcal{F} = \phi^*(\mathcal{L}_\lambda)$ *for some* $\lambda \in \mathbf{C}\backslash\mathbf{R}$.

Furthermore ϕ is injective as a map between spaces of leaves, i.e. for every leaf $L \in \mathcal{L}_\lambda$ the preimage $\phi^{-1}(\phi(\overline{\Omega}) \cap L)$ is exactly one leaf of $\mathcal{F}|_{\overline{\Omega}}$.

References

1. M. Brunella and P. Sad, *Holomorphic foliations in certain holomorphically convex domains of* $\mathbf{C}^2$. Bull. Soc. Math. France **123** (1995), 535–546.
2. C. Camacho, N. Kuiper and J. Palis, *The topology of holomorphic flows with singularity*, Publ. Math. I.H.E.S. **48** (1978), 5–38.
3. C. Camacho and P. Sad, *Invariant varieties through singularities of holomorphic vector fields*, Ann. of Math. **115** (1982), 579–595
4. H. Dulac, *Solutions d'un système d'équations différentielles dans le voisinage de valeurs singulières*, Bull. Soc. Math. France, **40** (1912), 324–383.
5. T. Ito, *A Poincaré-Bendixson type theorem for holomorphic vector fields*, RIMS Kokyuroku **878** (June, 1994), 1–9.
6. H. Poincaré, *Propiétés des fonctions définies par les équations aux différences partielles*, Thèse, Paris, 1879.

Department of Natural Sciences, Ryukoku University, Fushimi-ku, Kyoto 612, Japan